

The Commutation Relation for Cavity Mode Operators

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We have obtained the commutation relation for a cavity mode driven by coherent light and interacting with a two-level atom. We have found that the commutation relation for a free cavity mode with or without photons is just the usual commutation relation. However, the commutation relation for a cavity mode, which is interacting with a two-level atom, turns out to be different from the usual one.

It appears that the usual commutation relation $[\hat{a}, \hat{a}] = 1$ [1] is taken to be the commutation relation for a cavity mode regardless of whether the cavity mode is interacting with an atom or not. In order to check the validity of this viewpoint, we seek here to obtain the commutation relation for a cavity mode driven by coherent light and interacting with a two-level atom. The interaction between the cavity mode and the driving coherent light can be described by the Hamiltonian

$$\hat{H} = i\varepsilon(\hat{a}^\dagger - \hat{a}), \quad (1)$$

with ε being proportional to the amplitude of the driving coherent light. In addition, the interaction between the cavity mode and the two-level atom can be described at resonance by the Hamiltonian [2]

$$\hat{H} = ig(\hat{\sigma}^\dagger \hat{a} - \hat{a}^\dagger \hat{\sigma}), \quad (2)$$

where

$$\hat{\sigma} = |b\rangle\langle a| \quad (3)$$

is a lowering atomic operator, \hat{a} is the annihilation operator for the cavity mode, and g is the coupling constant between the atom and the cavity mode. Here $|a\rangle$ and $|b\rangle$ are the upper and lower levels of the two-level atom. We thus see that the Hamiltonian describing the interaction of the cavity mode with the driving coherent light and the two-level atom has the form

$$\hat{H} = i\varepsilon(\hat{a}^\dagger - \hat{a}) + ig(\hat{\sigma}^\dagger \hat{a} - \hat{a}^\dagger \hat{\sigma}). \quad (4)$$

We consider the case in which the cavity mode is coupled to a vacuum reservoir via a single-port mirror. In addition, we carry out our calculation by taking into account the noise operators associated with the vacuum reservoir. The quantum Langevin equation for the operator \hat{a} is given by [3]

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - i[\hat{a}, \hat{H}] + \hat{F}(t), \quad (5)$$

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where κ is the cavity damping constant and $\hat{F}(t)$ is a noise operator with vanishing mean. Now with the aid of (4), we easily find

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - g\hat{\sigma} + \varepsilon + \hat{F}(t). \quad (6)$$

In addition, employing the relation

$$\frac{d}{dt}\langle\hat{A}\rangle = -i\langle[\hat{A}, \hat{H}]\rangle \quad (7)$$

along with Eq. (4), one readily obtains

$$\frac{d}{dt}\langle\hat{\sigma}\rangle = g\langle(\hat{\eta}_b - \hat{\eta}_a)a\rangle, \quad (8)$$

$$\frac{d}{dt}\langle\hat{\eta}_a\rangle = g\langle\hat{\sigma}^\dagger\hat{a}\rangle + g\langle\hat{a}^\dagger\hat{\sigma}\rangle, \quad (9)$$

where

$$\hat{\eta}_a = |a\rangle\langle a|, \quad (10)$$

$$\hat{\eta}_b = |b\rangle\langle b|. \quad (11)$$

We see that Eqs. (8) and (9) are nonlinear differential equations and hence it is not possible to obtain exact solutions of these equations. We seek to overcome this problem by making use of the so-called large-time approximation scheme [4]. Thus applying this approximation scheme, we obtain from Eq. (6) the approximately valid relation

$$\hat{a}(t) = -\frac{2g}{\kappa}\hat{\sigma}(t) + \frac{2\varepsilon}{\kappa} + \frac{2}{\kappa}\hat{F}(t). \quad (12)$$

Now substitution of (12) (with the time argument suppressed) into the aforementioned equations yields

$$\frac{d}{dt}\langle\hat{\sigma}\rangle = -\frac{1}{2}\gamma_c\langle\hat{\sigma}\rangle + \frac{2g\varepsilon}{\kappa}\langle\hat{\eta}_b - \hat{\eta}_a\rangle + \frac{2g}{\kappa}\langle(\hat{\eta}_b - \hat{\eta}_a)\hat{F}\rangle, \quad (13)$$

$$\frac{d}{dt}\langle\hat{\eta}_a\rangle = -\gamma_c\langle\hat{\eta}_a\rangle + \frac{2g\varepsilon}{\kappa}\langle\hat{\sigma} + \hat{\sigma}^\dagger\rangle + \frac{2g}{\kappa}\langle\hat{\sigma}^\dagger\hat{F} + \hat{F}^\dagger\hat{\sigma}\rangle, \quad (14)$$

where

$$\gamma_c = 4g^2/\kappa. \quad (15)$$

Assuming that the atomic and noise operators are not correlated, we have

$$\langle(\hat{\eta}_b - \hat{\eta}_a)\hat{F}\rangle = \langle\hat{\eta}_b - \hat{\eta}_a\rangle\langle\hat{F}\rangle = 0, \quad (16)$$

$$\langle\hat{\sigma}^\dagger\hat{F}\rangle = \langle\hat{\sigma}^\dagger\rangle\langle\hat{F}\rangle = 0, \quad (17)$$

$$\langle\hat{F}^\dagger\hat{\sigma}\rangle = \langle\hat{F}^\dagger\rangle\langle\hat{\sigma}\rangle = 0. \quad (18)$$

We therefore see that

$$\frac{d}{dt}\langle\hat{\sigma}\rangle = -\frac{1}{2}\gamma_c\langle\hat{\sigma}\rangle + \frac{2g\varepsilon}{\kappa}\langle\hat{\eta}_b - \hat{\eta}_a\rangle. \quad (19)$$

$$\frac{d}{dt}\langle\hat{\eta}_a\rangle = -\gamma_c\langle\hat{\eta}_a\rangle + \frac{2g\varepsilon}{\kappa}\langle\hat{\sigma} + \hat{\sigma}^\dagger\rangle. \quad (20)$$

Moreover, we note that the steady-state solutions of Eqs. (19) and (20) have the form

$$\langle\hat{\sigma}\rangle = \frac{4g\varepsilon}{\kappa\gamma_c}\langle\hat{\eta}_b - \hat{\eta}_a\rangle, \quad (21)$$

$$\langle \hat{\eta}_a \rangle = \frac{2g\varepsilon}{\kappa\gamma_c} \langle \hat{\sigma} + \hat{\sigma}^\dagger \rangle. \quad (22)$$

Hence on substituting (21) into Eq. (22), we find

$$\langle \hat{\eta}_a \rangle = \frac{4\varepsilon^2}{\kappa\gamma_c} \langle \hat{\eta}_b - \hat{\eta}_a \rangle. \quad (23)$$

Now taking into account the completeness relation

$$\hat{\eta}_a + \hat{\eta}_b = \hat{I}, \quad (24)$$

we get

$$\langle \hat{\eta}_a \rangle = \frac{4\varepsilon^2}{8\varepsilon^2 + \kappa\gamma_c} \quad (25)$$

and applying once more Eq. (24), we have

$$\langle \hat{\eta}_b \rangle = 1 - \frac{4\varepsilon^2}{8\varepsilon^2 + \kappa\gamma_c}. \quad (26)$$

It is not hard to realize that $\langle \hat{\eta}_a \rangle$ and $\langle \hat{\eta}_b \rangle$ represent the probabilities for the two-level atom to be in the upper and lower levels, respectively. Finally, with the aid of (25) and (26), one can put Eq. (21) in the form

$$\langle \hat{\sigma} \rangle = \frac{4g\varepsilon}{8\varepsilon^2 + \kappa\gamma_c}. \quad (27)$$

Furthermore, applying the relation

$$\frac{d}{dt} \langle \hat{a}(t) \hat{a}^\dagger(t) \rangle = \left\langle \frac{d\hat{a}(t)}{dt} \hat{a}^\dagger(t) \right\rangle + \left\langle \hat{a}(t) \frac{d\hat{a}^\dagger(t)}{dt} \right\rangle \quad (28)$$

along with Eq. (6) and its adjoint, we readily get

$$\begin{aligned} \frac{d}{dt} \langle \hat{a}(t) \hat{a}^\dagger(t) \rangle = & -\kappa \langle \hat{a}(t) \hat{a}^\dagger(t) \rangle + \varepsilon \langle \hat{a}(t) + \hat{a}^\dagger(t) \rangle - g [\langle \hat{\sigma}(t) \hat{a}^\dagger(t) \rangle + \langle \hat{a}(t) \hat{\sigma}^\dagger(t) \rangle \\ & + \langle \hat{F}(t) \hat{a}^\dagger(t) \rangle + \langle \hat{a}(t) \hat{F}^\dagger(t) \rangle]. \end{aligned} \quad (29)$$

It proves to be convenient to replace the operators $\hat{a}^\dagger(t)$ and $\hat{a}(t)$ that appear in the second, third, fourth, and fifth terms of Eq. (29) by expression (12) and its adjoint. We then find

$$\begin{aligned} \frac{d}{dt} \langle \hat{a}(t) \hat{a}^\dagger(t) \rangle = & -\kappa \langle \hat{a}(t) \hat{a}^\dagger(t) \rangle + \gamma_c \langle \hat{\eta}_b \rangle + \frac{4\varepsilon^2}{\kappa} - \frac{4g\varepsilon}{\kappa} \langle \hat{\sigma}(t) + \hat{\sigma}^\dagger(t) \rangle \\ & - \frac{2g}{\kappa} (\langle \hat{\sigma}(t) \hat{F}^\dagger(t) \rangle + \langle \hat{F}(t) \hat{\sigma}^\dagger(t) \rangle) + \langle \hat{F}(t) \hat{a}^\dagger(t) \rangle + \langle \hat{a}(t) \hat{F}^\dagger(t) \rangle, \end{aligned} \quad (30)$$

so that in view of the assumption that the atomic and noise operators are not correlated, there follows

$$\begin{aligned} \frac{d}{dt} \langle \hat{a}(t) \hat{a}^\dagger(t) \rangle = & -\kappa \langle \hat{a}(t) \hat{a}^\dagger(t) \rangle + \gamma_c \langle \hat{\eta}_b \rangle + \frac{4\varepsilon^2}{\kappa} - \frac{4g\varepsilon}{\kappa} \langle \hat{\sigma}(t) + \hat{\sigma}^\dagger(t) \rangle \\ & + \langle \hat{F}(t) \hat{a}^\dagger(t) \rangle + \langle \hat{a}(t) \hat{F}^\dagger(t) \rangle. \end{aligned} \quad (31)$$

Moreover, the solution of Eq. (6) can be written as

$$\hat{a}(t) = \hat{a}(0)e^{-\kappa t/2} + e^{-\kappa t/2} \int_0^t e^{\kappa t'/2} [-g\hat{\sigma}(t') + \varepsilon + \hat{F}(t')] dt', \quad (32)$$

and multiplying this equation on the right by $\hat{F}^\dagger(t)$, we have

$$\langle \hat{a}(t)\hat{F}^\dagger(t) \rangle = \langle \hat{a}(0)\hat{F}^\dagger(t) \rangle e^{-\kappa t/2} + e^{-\kappa t/2} \int_0^t e^{\kappa t'/2} [-g\langle \hat{\sigma}(t')\hat{F}^\dagger(t) \rangle + \langle \hat{F}(t')\hat{F}^\dagger(t) \rangle] dt', \quad (33)$$

in which we have used the fact that $\varepsilon\langle \hat{F}^\dagger(t) \rangle = 0$. Now in view of the assumptions

$$\langle \hat{a}(0)\hat{F}^\dagger(t) \rangle = 0 \quad (34)$$

and

$$\langle \hat{\sigma}(t')\hat{F}^\dagger(t) \rangle = 0, \quad (35)$$

Eq. (32) takes the form

$$\langle \hat{a}(t)\hat{F}^\dagger(t) \rangle = e^{-\kappa t/2} \int_0^t e^{\kappa t'/2} \langle \hat{F}(t')\hat{F}^\dagger(t) \rangle dt'. \quad (36)$$

Hence using the correlation property [5]

$$\langle \hat{F}(t')\hat{F}^\dagger(t) \rangle = \kappa\delta(t-t'), \quad (37)$$

we arrive at

$$\langle \hat{a}(t)\hat{F}^\dagger(t) \rangle = \frac{\kappa}{2}. \quad (38)$$

Therefore, on substituting (38) and its complex conjugate into Eq. (31), there follows

$$\frac{d}{dt}\langle \hat{a}(t)\hat{a}^\dagger(t) \rangle = -\kappa\langle \hat{a}(t)\hat{a}^\dagger(t) \rangle + \gamma_c\langle \hat{\eta}_b \rangle + \frac{4\varepsilon^2}{\kappa} - \frac{4g\varepsilon}{\kappa}\langle \hat{\sigma}(t) + \hat{\sigma}^\dagger(t) \rangle + \kappa. \quad (39)$$

The steady-state solution of Eq. (39) has the form

$$\langle \hat{a}\hat{a}^\dagger \rangle = \frac{\gamma_c}{\kappa}\langle \hat{\eta}_b \rangle + \frac{4\varepsilon^2}{\kappa^2} - \frac{4g\varepsilon}{\kappa^2}\langle \hat{\sigma} + \hat{\sigma}^\dagger \rangle + 1 \quad (40)$$

and on taking into account (27), we obtain

$$\langle \hat{a}\hat{a}^\dagger \rangle = \frac{\gamma_c}{\kappa}\langle \hat{\eta}_b \rangle + \frac{4\varepsilon^2}{\kappa^2} - \frac{\gamma_c}{\kappa} \frac{8\varepsilon^2}{8\varepsilon^2 + \kappa\gamma_c} + 1. \quad (41)$$

Following the same procedure, we can also readily establish that at steady state

$$\langle \hat{a}^\dagger\hat{a} \rangle = \frac{\gamma_c}{\kappa}\langle \hat{\eta}_a \rangle + \frac{4\varepsilon^2}{\kappa^2} - \frac{\gamma_c}{\kappa} \frac{8\varepsilon^2}{8\varepsilon^2 + \kappa\gamma_c}. \quad (42)$$

Now employing Eqs. (41) and (42), we easily find

$$[\hat{a}, \hat{a}^\dagger] = \frac{\gamma_c}{\kappa}(\langle \hat{\eta}_b \rangle - \langle \hat{\eta}_a \rangle) + 1. \quad (43)$$

Finally, with the aid of Eqs. (25) and (26), one can put Eq. (43) in the form

$$[\hat{a}, \hat{a}^\dagger] = \frac{\gamma_c^2}{8\varepsilon^2 + \kappa\gamma_c} + 1 \quad (44)$$

and on account of (25) the mean photon number of the cavity mode has the form

$$\bar{n} = \frac{4\varepsilon^2}{\kappa^2} - \frac{\gamma_c}{\kappa} \frac{4\varepsilon^2}{8\varepsilon^2 + \kappa\gamma_c}. \quad (45)$$

We notice that Eq. (44) represents the commutation relation for a cavity mode which is interacting with a two-level atom. We notice that the first term on the right side of this equation is due to the interaction of the cavity mode with the two-level atom and the second term is due to the vacuum reservoir noise. We now consider the case in which the cavity mode is not interacting with the atom. Then for this case ($g=0$), Eq. (44) takes the form

$$[\hat{a}, \hat{a}^\dagger] = 1, \quad (46)$$

with

$$\bar{n} = \frac{4\varepsilon^2}{\kappa^2}. \quad (47)$$

This represents the commutation relation for a free cavity mode with photons. Alternatively, in the absence of the driving coherent light ($\varepsilon = 0$), Eq. (44) has the form

$$[\hat{a}, \hat{a}^\dagger] = \frac{\gamma_c}{\kappa} + 1, \quad (48)$$

with $\bar{n} = 0$. We see from Eq. (26) that the atom in this case is in the lower level. We observe that Eq. (48) represents the commutation relation for a vacuum cavity mode which is interacting with a two-level atom. We may envisage this interaction as a process in which virtual photons are absorbed and emitted, with the atom continuing to be in the lower level. We note that in the absence of this interaction ($g=0$), Eq. (48) goes over into

$$[\hat{a}, \hat{a}^\dagger] = 1. \quad (49)$$

We now realize that the usual commutation relation is just the commutation relation for a free cavity mode with or without photons.

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